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# A CRITERION FOR THE NON-EXISTENCE OR NON-UNIQUENESS OF SOLUTIONS OF SELF-SIMILAR PROBLEMS IN THE MECHANICS OF A CONTINUOUS MEDIUM†

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Systems of hyperbolic partial differential equations expressing conservation laws are considered. A sufficient condition is formulated under which the self-similar problem of the disintegration of an arbitrary discontinuity (or the "piston" problem) either has no solution or the solution is not unique. This sufficient condition is determined by the existence of non-evolutionary discontinuities which may be considered as a sequence of two evolutionary discontinuities moving at the same velocity, if such a representation is unique. The condition is more general than that formulated previously, which was based on the existence of a non-proper Jouguet point. The new criterion is satisfied by weak quasitranverse shock waves in elastic media, whatever the sign of the coefficient of the non-linear deformation term. It also enables one to draw conclusions as to the non-existence or non-uniqueness of solutions of problems of the theory of elasticity in the case of finite-amplitude waves. © 2002 Elsevier Science Ltd. All rights reserved.

## 1. INTRODUCTION

Consider a system of quasilinear homogeneous hyperbolic equations obtained from integral conservation laws for the one-dimensional unsteady case

$$\frac{\partial f_k(u_i(x,t))}{\partial t}+\frac{\partial g_k(u_i(x,t))}{\partial x}=0,\quad i,k=1,2,\dots,n \tag{1.1}$$

The functions $f_k$ may be treated as the densities of certain physical quantities, $g_k$ are the fluxes of these quantities through an element of unit area perpendicular to the $x$ axis, and $u_i$ are the unknown state parameters. The hyperbolic system (1.1) admits of $n$ real characteristic velocities $c_1(u_i),\dots,c_n(u_i)$. Corresponding to these are $n$ continuous solutions – Riemann waves, which may tilt, forming discontinuities.

The relations at the discontinuities are obtained from the same integral conservation laws, and they have the form

$$[g_k]-W[f_k]=0,\quad k=1,2,\dots,n \tag{1.2}$$

where $W$ is the velocity of the discontinuity, and $[f]=f^+-f^-$ the difference between the post- and pre-jump values of $f$. The relations at the discontinuity make it possible, for every pre-jump state $u_i^-=U_i$ and every value of $W$, to specify the possible post-jump states $u_i^+$. Elimination of $W$ from the relations yields the shock adiabat – a curve in the state $u_i$-space passing through the initial point $U_i$ and intersecting itself at that point, since the branches of the shock adiabat are tangent at that point to the integral curves of $n$ Riemann waves [1]. Using the same Eqs (1.2), one can then find the velocity $W$ of the jump as a function of the position of the point on the shock adiabat.

The problem of the disintegration of an arbitrary initial discontinuity for system (1.1) is formulated as follows: at $t=0$, the plane $x=0$ separates two half-planes with homogeneous states: $x>0$, $u_i=U_i=\text{const}$ and $x<0$, $u_i=u_i^*=\text{const}$. At $t>0$ sequences of self-similar plane waves propagate from the boundary in both directions, as discontinuity fronts and centred Riemann waves, which are to be determined. One can also consider the self-similar problem in the half-space $x>0$ only, specifying constant values $u_i=u_i^*$ at the boundary ("piston" problem).

Besides the equations of the shock adiabat, discontinuities that can be used in the solution must also satisfy the non-decreasing entropy condition and the evolution conditions [1, 2]. The evolution conditions are criteria ensuring that the system of homogeneous algebraic equations obtained by linearizing the

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conservation laws at the discontinuity (1.2) will be uniquely solvable for the amplitudes of small homogeneous perturbations propagating from the discontinuity. The evolution conditions are expressed as systems of inequalities between the jump velocity $W$ and the characteristic velocities $c_j^-$, $c_k^+$, $(j, k = 1, 2, \ldots, n)$ before and after the discontinuity, respectively:

$$c_k^- \leq W \leq c_{k+1}^-, \quad c_{k-1}^+ \leq W \leq c_k^+, \quad k = 1, 2, \ldots, n \tag{1.3}$$

Inequalities (1.3) express the condition that the number of different types of perturbations propagating from the discontinuity (or, what is the same, the number of outgoing characteristics) should equal $n - 1$. When that is the case, the $n$ linearized relations (1.2) at the discontinuity must uniquely define the amplitudes of the $n - 1$ outgoing waves and the perturbation of the shock wave velocity.

For every $k$, such a pair of inequalities defines a type of discontinuity, which we will refer to as type-$k$ discontinuities. Obviously $k$ discontinuity types exist, depending on the number $k$ (inequalities involving the non-existent $c_0^+$ and $c_{n+1}^-$ must be omitted from conditions (1.3)).

Inequalities (1.3) define segments on the shock adiabat corresponding to evolutionary shock waves. At the endpoints of these segments the jump velocity is identical with one of the characteristic velocities, depending on the pre-jump state (if $W = c_k^-$) or the post-jump state (if $W = c_k^+$). These points (except the initial point) are known as Jouguet points.

For an intuitive representation of inequalities (1.3), it is convenient to use a diagram in which the velocities participating in inequalities (1.3) are plotted along mutually orthogonal axes, thus mapping the validity of these inequalities [3]. Each of the hatched rectangles in Fig. 1 corresponds to a certain type (the $k$th) of shock waves. Since it is known how the velocities $W$ and $c_k^+$ vary along the shock adiabat (the velocities $c_j^-$ do not vary), one can map the trace of the shock adiabat in the diagram as a function of $W$ along itself. The velocity of an infinitely weak $k$th discontinuity, i.e., the velocity of the jump at the initial point $A$, is identical with the pre- and post-jump characteristic velocities $W = c_k^- = c_k^+$. This means that the initial state $A(c_k^-, c_k^+)$ $(k = 1, 2, \ldots, n)$ is represented in the diagram of Fig. 1 by $n$ lattice points.

Lax formula [1] for the velocity of weak shock waves, $W = (c_k^- + c_k^+)/2$, enables one to conclude that, in the neighbourhood of the above-mentioned lattice points, the shock adiabat through them goes off at one end into an evolution rectangle, and at the other end – into the "symmetric" non-evolution rectangle, as shown in Fig. 1. Thus the shock adiabat always contains at least one evolution segment for each of the $n$ types of discontinuity (and possibly more than one).

The linearized conservation laws at the discontinuity (1.2) are not sufficient for discontinuities represented by rectangles above the hatched ones in Fig. 1 to have the evolution property. To make such shock waves evolutionary, one can expand system (1.2) by adding relations independent of the conservation laws.

In what follows, we will be particularly interested in the discontinuities represented in Fig. 1 by rectangles adjoining evolutionary rectangles below and on the right. For such discontinuities to be evolutionary, the number of boundary conditions must equal $n - 1$, while in fact the number of relations (1.2) is $n$, that is, one more than necessary for these discontinuities to be evolutionary. Such

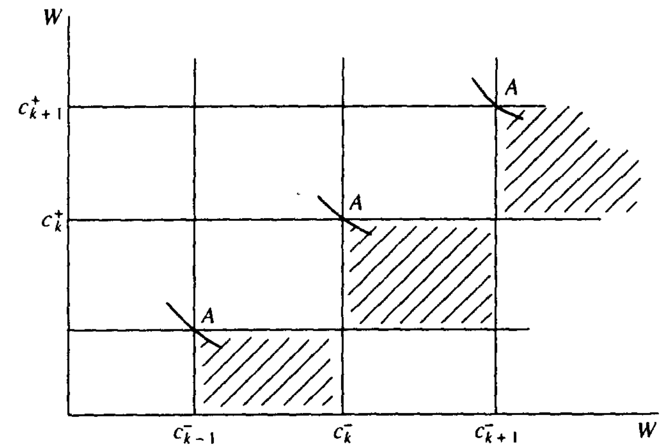


Fig. 1

discontinuities will henceforth be called non-evolutionary {1} (the number in braces is the number of superfluous boundary conditions).

We will show that the evolution conditions do not contradict the formal possibility of replacing such a non-evolutionary discontinuity by two evolutionary discontinuities moving at the same velocity. Indeed, in the region between these discontinuities there are $n$ distinct types of small perturbations, that is, all possible types of perturbation, each of which propagates from either one or the other discontinuity. There are $n-2$ perturbations that propagate from the combination of two discontinuities into the exterior region, just as from the non-evolutionary discontinuity that they are replacing. Thus, the total number of perturbations propagating from the two discontinuities is $2n-2$. If the same number of perturbations propagates from each, namely $n-1$, then both discontinuities are evolutionary. Whether in fact a non-evolutionary discontinuity may be represented by a sequence of two evolutionary discontinuities (we shall call this disintegration) depends on the properties of the shock adiabat; this question will be considered later.

Note that a non-evolutionary {1} discontinuity may only disintegrate into two evolutionary discontinuities moving at the same velocity, since if it were to disintegrate into a larger number of evolutionary discontinuities, the relations following from the conservation laws would not uniquely define the amplitudes of small perturbations and perturbations of the velocities of the discontinuities. In addition, it follows from the above arguments that, if one of the two discontinuities into which a non-evolutionary {1} discontinuity disintegrates is evolutionary, the other is also evolutionary.

## 2. THE DISINTEGRATION OF NON-EVOLUTIONARY {1} DISCONTINUITIES AND THE UNIQUENESS OF THE SOLUTIONS

When the shock adiabat is mapped on the evolution diagram of Fig. 1 (for a fixed pre-shock-wave state), the quantities $c_i^-$ are determined by the initial state and are therefore constant. Hence the velocity $W$ can be mapped along the horizontal axis in some fixed scale. This cannot be done along the vertical axis, because the quantities $c_i^+$ may vary along the shock adiabat. Points of the shock adiabat lying on the same vertical line in the diagram then correspond to discontinuities moving at the same velocity with the same initial state.

Consider the situation illustrated in Fig. 2: an evolutionary segment above a non-evolutionary {1} segment of the shock adiabat with initial point $A$ (only parts of the shock adiabat are shown in the figure). Consider points $M$ and $M_1$ of the shock adiabat corresponding to the same velocity $W$. The corresponding points on the shock adiabat in the phase $U_i$-space will be denoted by the same letters $M$ and $M_1$. Since the discontinuities $A \to M$ and $A \to M_1$ (where $A$ is the initial point of the shock adiabat) are moving at the same velocity $W$, it follows that if the conservation laws (1.2) are satisfied, the conservation laws will also be satisfied at the discontinuity $M_1 \to M$, which is moving at the same velocity $W$. Thus, the discontinuity $A \to M$ disintegrates, that is, a sequence of discontinuities $A \to M_1$ and $M_1 \to M$ exists, each of which is moving at the same velocity $W$ as the non-evolutionary {1} discontinuity $A \to M$, while their pre-jump ($A$) and post-jump states ($M$) are the same. By our previous discussion, the evolution property for the discontinuity $A \to M_1$, which is of type $k$, implies the same condition for the discontinuity

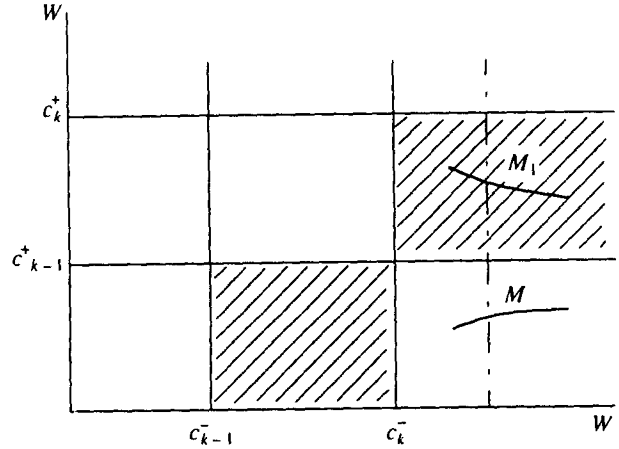


Fig. 2

$M_1 \to M$, which is obviously of type $k-1$. Throughout what follows, when referring to shock adiabats, we shall indicate their initial points in parentheses.

We will now consider a series of problems concerning the disintegration of an arbitrary discontinuity (or "piston" problems) in which all the waves (which may be discontinuities or Riemann waves), apart from those numbered $k$ and $k-1$, are fixed. The state before the $k$th wave $(A)$ will be fixed and taken as the initial state for the shock adiabat $(A)$. Let us find the set of points in $u_i$-space that can be reached from $A$ by a sequence of two discontinuities of types $k$ and $k-1$. By the previous reasoning, this set also contains the non-evolutionary $\{1\}$ part of the shock adiabat $(A)$ for whose points, at the same $W$ values, evolutionary discontinuities of type $k$ exist (in the diagram of Fig. 2 this is the part of the non-evolutionary $\{1\}$ section of the shock adiabat above which the evolutionary segment is situated). Along this non-evolutionary $\{1\}$ part of the shock adiabat $(A)$, the velocities of the $k$th and $(k-1)$-st discontinuities are identical. In a self-similar solution, obviously, the $(k-1)$-st discontinuity should have a velocity not exceeding that of the $k$th, $W_{k-1} \leqslant W_k$. If we fix the point $M_1$, and consequently $W_k$ also, then, as $W_{k-1}$ varies, the point representing the state after the sequence of the $k$th and $(k-1)$-st discontinuities will move along a shock adiabat with initial point $M_1$. When the velocity $W_{k-1}$ reaches the values $W_k$, the state $(M)$ after the system of two discontinuities moving at the same velocity must belong to a non-evolutionary $\{1\}$ segment of the shock adiabat. Let us consider the general situation, when the velocity $W_{k-1}$ does not have extremum on the shock adiabat $(M_1)$ at the point $M$ where it intersects the shock adiabat $(A)$.

As the point $M_1$ moves along the shock adiabat $(A)$, the $(k-1)$st shock adiabat $(M_1)$ in $u_i$-space sweeps out a two-dimensional surface. Since in a self-similar problem it must be true that $W_{k-1} \leqslant W_k$, the state after the two discontinuities, of types $k$ and $k-1$, may belong only to the part of that surface bounded by the non-evolutionary $\{1\}$ section of the shock adiabat $(A)$.

If we now decide to vary the $n-2$ waves which were previously considered to be fixed, the curve corresponding to the non-evolutionary $\{1\}$ part of the shock adiabat will – in the case of the general position – sweep out in $u_i$-space a surface (hypersurface) $\Sigma$ of dimension $n-1$. This surface divides $u_i$-space into two parts, in one of which the solution of the disintegration problem contains shock waves of types $k$ and $k-1$ whose velocities are the same on the surface $\Sigma$ and equal the velocity of the non-evolutionary $\{1\}$ discontinuity. We shall refer to this solution as solution I.

If only one pair of discontinuities of types $k$ and $k-1$ exists with equal velocities into which the non-evolutionary discontinuity disintegrates, then it is obvious that, on the other side of the surface $\Sigma$, the solution of the problem will not simultaneously contain discontinuities of types $k$ and $k-1$ whose velocities are identical on $\Sigma$. Hence, if a solution in the region adjoining $\Sigma$ from the other side exists, it will contain either (at least) one of the Riemann waves of types $k$ and $k-1$, or discontinuities of types $k$ and $k-1$ with different velocities on the surface $\Sigma$. Such solutions (if they exist) will be called solutions II. The regions in which solutions II exist in the general case are not connected with the shock adiabat $(A)$ and are therefore not bounded by the surface $\Sigma$.

Indeed, the constraints on the region of existence of solutions II have nothing to do with the possibility of whether one non-evolutionary jump exists, which satisfies the conservation laws, from the state before the $k$th wave to a point of the surface $\Sigma$, since in these solutions combinations of type $k$ and $k-1$ waves do not have the same velocity. If a solution II exists on the same side of the surface $\Sigma$ where there are no solutions I, and its region of existence has nothing to do with $\Sigma$, then in the general case it may either not reach $\Sigma$ – which implies that there is no solution in some zone – or a region must exist in which there are solutions of both types I and II, that is, a region in which the solution is not unique.

Thus, if the initial point for the shock adiabat is suitably chosen, so that it has a non-evolutionary $\{1\}$ segment such that, in a certain velocity range, both non-evolutionary and evolutionary discontinuities exist, but only one of the latter, then one can also formulate conditions for the problem of the disintegration of discontinuities under which there is either no solution or the solution is not unique.

The above statement about the non-existence or non-uniqueness of solutions becomes meaningless if there are no evolutionary discontinuities with the same velocities as non-evolutionary $\{1\}$ ones, or if there are two such evolutionary discontinuities for each velocity. In the first case, the possible solutions have nothing whatever to do with the presence of a non-evolutionary wave of the type considered, and no conclusions as to solutions can be drawn in the manner proposed above. In the second case, the same hatched ($k$th) rectangle in the diagram of Fig. 2 will contain yet another evolutionary segment of the shock adiabat $(A)$. Then two combinations of $k$th and $(k-1)$st evolutionary waves will exist in the case when the remaining $n-2$ waves have fixed amplitudes. If one now allows these amplitudes to vary arbitrarily, then the segment of the shock adiabat corresponding to non-evolutionary $\{1\}$ discontinuities will, as before, sweep out an $(n-1)$-dimensional surface $\Sigma$. But now it may turn out that, on one side of that surface, there is a solution I containing one combination of $k$th and $(k-1)$st discontinuities, and, on the other side, a solution II containing another combination of discontinuities of those types, with the surface $\Sigma$ separating the regions corresponding to solutions I and solutions II. In this case, therefore, solutions containing discontinuities of types $k$ and $k-1$ exist on either side of the surface $\Sigma$ and are uniquely defined. It is, of course, possible that the solutions I and II correspond to regions bordering on the surface $\Sigma$ on the same side. When that happens, of course, the solution is not unique. It does not seem possible to distinguish between this case and the previous one on the sole basis of the form of the shock adiabat in the diagram.

If there are three or more evolutionary discontinuities having the same velocity as a non-evolutionary $\{1\}$ discontinuity, then on at least one side of the surface $\Sigma$ two or more solutions containing discontinuities of types $k$ and $k-1$ will obviously exist, so that the solution will not be unique.

Let us consider the case in which a solution exists but is not unique. Non-uniqueness is due to the appearance of a solution (designated above as a solution II) not directly related to the shock adiabat, defined in a certain finite region whose interior contains a non-evolutionary $\{1\}$ segment of the shock adiabat. Therefore, if such a non-evolutionary segment adjoins an evolutionary segment on the shock adiabat, part of that evolutionary segment will enter the existence region of the solution II which, as observed previously, consists of waves moving at different velocities. This means that shock waves corresponding to points of evolutionary segments of the shock adiabat which adjoin non-evolutionary $\{1\}$ segments may be replaced by a system of waves with the same parameter values before and after the wave system. In these cases, the equations expressing the conservation laws admit of disintegration of the above-mentioned evolutionary discontinuities into a system of waves (which may contain other evolutionary discontinuities).

The determination of the conditions under which such a disintegration of an evolutionary shock wave may in fact occur needs special investigation in specific cases. This question will not be considered here.

Such evolutionary discontinuities, which may disintegrate into a system of waves, have been observed in the investigation of quasitransverse waves in an anisotropic elastic medium [4]. However, as further research has shown [5, 6], the perturbations must be strong (strong incident waves or a variation of the background) for such disintegration to occur.

## 3. A REMARK ON THE STRUCTURE OF A NON-EVOLUTIONARY {1} DISCONTINUITY

As has been shown [7, 8], if the number of conservation laws exceeds the number $q$ of relations necessary for an evolutionary discontinuity, the structure of the discontinuity will contain $q$ arbitrary parameters. In the case under consideration there is a single-parameter set of integral curves (ICs) joining the initial singular point ($A$) and the final singular point ($B$) of the system of equations describing the structure of the non-evolutionary $\{1\}$ discontinuity (it is assumed that dissipative processes are sufficiently diverse, so that the solution describing the structure is continuous). Let us consider the two-dimensional surface consisting of the ICs emerging from $A$ and arriving at $B$. If, as usual, the singularities of the field of ICs are the stationary singular points of the system with simple eigenvalues (the coordinates of these points together with those of the points $A$ and $B$ satisfy the conservation laws), then either the aforementioned set of ICs is not bounded at all, or it is bounded by the separatrices of singular points other than $A$ and $B$. Among these points there may be some point $C$ such that a discontinuity corresponding to the transition $A \to C$ is evolutionary. As remarked, the discontinuity $C \to B$ will also be evolutionary. A unique IC corresponds to evolutionary discontinuities. The behaviour of the ICs joining the points $A$ and $B$ is illustrated in the case of a single additional point ($C$) in Fig. 3(a), and in the case of two singular points ($E$ and $F$) in Fig. 3(b). Thus, the set of solutions representing the structure of a non-evolutionary $\{1\}$ discontinuity may contain solutions representing the sequence of two evolutionary discontinuities $A \to C$ and $C \to B$, corresponding to the situation discussed in Section 2.

The number of small perturbations of different types leaving from a non-evolutionary $\{1\}$ discontinuity is one less than that leaving from an evolutionary discontinuity. By analogy with problems of magnetohydrodynamics, for which numerical experiments have been carried out to study the behaviour of non-evolutionary discontinuities [9], the drop in the number of outgoing perturbations may be treated as the capture and accumulation of perturbations of a certain type within the structure by the non-evolutionary $\{1\}$ discontinuity. By analogy with these results, we may expect that in such a process, if the amplitude of perturbations reaching the discontinuity is small, there will be a quasi-stationary restructuring of the structure of the discontinuity, due to an exchange of ICs representing the structure, without a change in the positions of the points $A$ and $B$. If the incident perturbations do not vary in

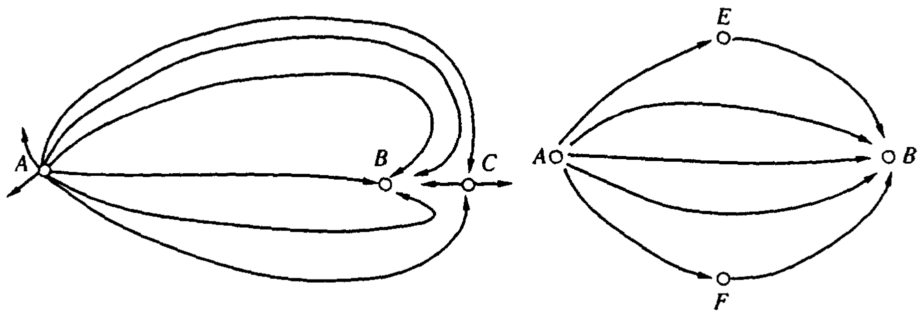


Fig. 3

time, their accumulation within the structure will lead to a monotonic shift of ICs in a certain direction, until they encounter the singular point $C$, so that ICs $A \to B$ disintegrate into two parts, $A \to C$ and $C \to B$, corresponding to discontinuities with the same velocities. With further action of the incident perturbations, these discontinuities take on different velocities. The "amount" of incident perturbations necessary for the disintegration of a non-evolutionary wave is less, the smaller the width of the transition zone of the structure of the discontinuity, that is, the smaller the dissipative coefficients. If the dissipative coefficients tend to zero, the action of arbitrarily small perturbations will suffice for a non-evolutionary discontinuity to disintegrate (as this is guaranteed by its non-evolutionary character).

## 4. SOME APPLICATIONS OF THE CRITERION OBTAINED IN SECTION 2

Let us consider the problem of an abrupt change in the load at the boundary of an elastic anisotropic half-space (the "piston" problem). A complete solution of this problem, which in some cases is non-unique, is a available for low-intensity non-linear elastic waves [4]. This enables us to illustrate the action of the criteria developed above in the case of low-amplitude quasitransverse waves and to extend the result to problems with finite-amplitude waves.

Our investigation will be carried out in Lagrange variables corresponding to the Cartesian coordinates of the initial state [10]. The $x = x_3$ axis is orthogonal to the boundary of the half-space $x > 0$, which is filled with the elastic medium in which the solution will be sought. The $x_1$ and $x_2$ axes are parallel to the plane of the wave front. The deformation is characterized by the gradients of the displacement $\partial w_i/\partial x$, of which only $\partial w_i/\partial x = u_i(x, t)$ $(i = 1, 2, 3)$ vary in the plane waves considered here.

The medium is defined by its elastic potential (the internal energy per unit volume of the medium prior to deformation) $\Phi(u_i, S)$, where $S$ is the entropy. Assuming that the non-linearity and anisotropy are small and taking only the principal terms responsible for these effects into consideration, we express the function $\Phi$ by its expansion in powers of $u_i$ and $\Delta S$. Considering only quasitransverse waves, we can introduce a certain effective incompressible medium in which $u_3$ does not vary, while $u_1$ and $u_2$ vary as in quasitransverse waves in the original medium [4]. For this effective medium, the aforementioned expansion of $\Phi$ is

$$\Phi = \frac{f}{2}(u_1^2 + u_2^2) + \frac{g}{2}(u_2^2 - u_1^2) - \frac{\varkappa}{4}(u_1^2 + u_2^2)^2 + \rho_0 T_0(S - S_0) \tag{4.1}$$

where $f$, $g$ and $\varkappa$ are constant coefficients. The first term defines the elastic potential of a linear isotropic medium, and the coefficient $f$ differs only slightly from the shear modulus. The second term expresses the anisotropy of the medium in the plane of the front (wave anisotropy), the coefficient $g$ may always be considered to be positive, and moreover $g \ll f$, implying the assumption that the anisotropy is small. The non-linearity of the medium is represented by the term with coefficient $\varkappa$, which may be positive or negative, depending on the direction in which the stress-strain graph is convex. The sign of $\varkappa$ is positive when the graph is convex upward and negative when it is convex downward.

In this approximation the system of equations of the theory of elasticity has the form

$$\frac{\partial v_\alpha}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial u_\alpha}\right), \quad \frac{\partial v_\alpha}{\partial x} = \frac{\partial u_\alpha}{\partial t}, \quad v_\alpha = \frac{\partial w_\alpha}{\partial t}, \quad \alpha = 1,2 \tag{4.2}$$

which has two pairs of characteristic velocities $\pm c_1$ and $\pm c_2$ corresponding to quasitransverse waves. We shall consider only waves propagating into the half-space $x > 0$ filled with the medium, taking $c_1(u_\alpha) \leqslant c_2(u_\alpha)$.

The conservation laws at the discontinuity have the form

$$\left[\frac{\partial \Phi}{\partial u_a}\right] = W^2[u_\alpha], \quad \alpha = 1,2 \tag{4.3}$$

In the case of shock waves (SWs) they make it possible, by elimination of the jump velocity $W$, to obtain the shock adiabat for given $u_\alpha^- = U_\alpha$ – a curve in the phase space $(u_1, u_2)$ which intersects itself at the initial point $A(U_\alpha)$. The same relations (4.3) will yield the jump velocity $W$ along the shock adiabat and enable us to formulate the evolution conditions (1.3).

(a) $c_1^- \leqslant W \leqslant c_2^-$, $\quad 0 < W \leqslant c_1^+$ – slow SWs,

(b) $c_2^- \leqslant W \quad c_1^+ \leqslant W \leqslant c_2^+$ – fast SWs.

These inequalities single out the evolutionary segments on the shock adiabat, as illustrated in the diagram of Fig. 4 by the parts of the curves in the hatched rectangles. The shape of the shock adiabat and the configuration of the evolutionary segments in both on the phase plane $(u_1, u_2)$ and on the diagram depend on the quotients $g/(\varkappa(U_1^2 + U_2^2))$ and $U_1/U_2$, which are given by the formulation of the problem.

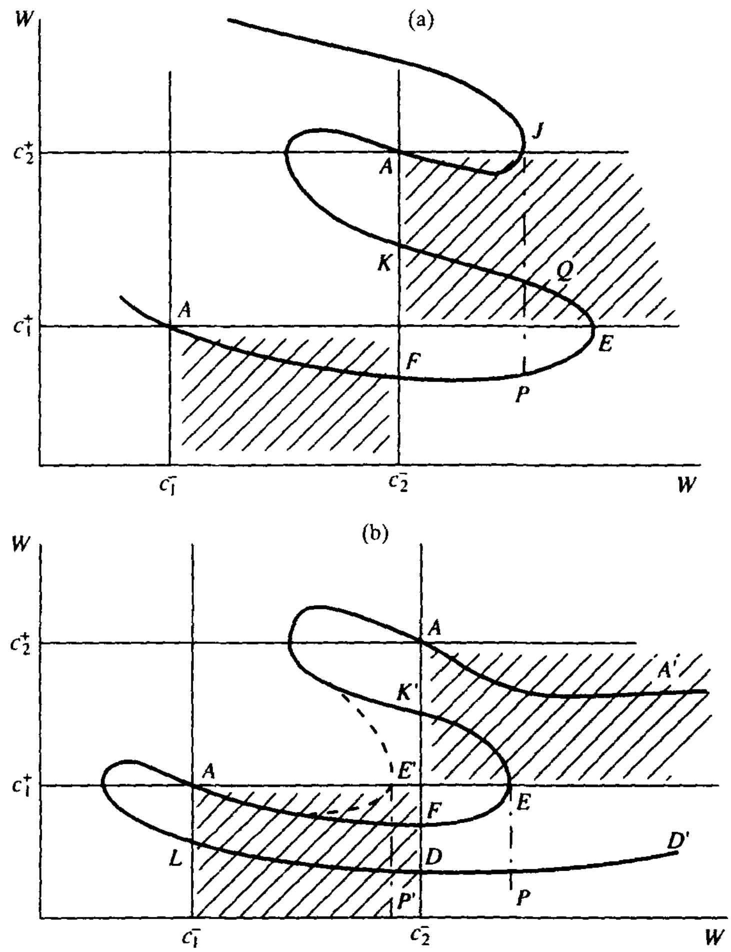


Fig. 4

We will first consider a medium for which $\varkappa > 0$. One possible position of the shock adiabat in the evolution diagram is shown in Fig 4(a). In the upper evolutionary rectangle, above the non-evolutionary segment $FP$ of the shock adiabat ($A$), there are two segments, corresponding to two fast evolutionary SWs. In the neighbourhood of the arc $FP$ in the $(u_1, u_2)$ plane a solution of the self-similar problem may be constructed, using each of these fast SWs followed by a slow SW. Direct solution of the problem shows that one of the solutions (type I) consists of the fast SW emerging from the point $A$ to state $M_1$ on the segment $KE$, and then a slow SW with shock adiabat ($M_1$). This solution acts in the $(u_1, u_2)$ plane in the region adjoining the arc $FP$ from above (Fig. 5a). On the arc $FP$ itself both SWs, fast and slow, have the same velocity and merge in the self-similar solution into a single non-evolutionary jump. On the other side of the arc $FP$ in the $(u_1, u_2)$ plane, there is another solution (type II), consisting of the sequence of a fast SW $A \to M_2$ (the point $M_2$ lies on the evolutionary part $AJ$ of the shock adiabat ($A$)) and then a slow SW from state $M_2$. On the arc $FP$ itself, the velocities of the fast and slow SWs in this solution are identical. Thus, the regions in which I- and II-solutions exist adjoin the arc $FP$, so that there is a unique solution for a whole neighbourhood of the arc $FP$.

Above the non-evolutionary segment $PE$ in the upper hatched rectangle in Fig. 4(a) there is only one evolutionary segment $QE$ of a shock adiabat ($A$). The criterion proposed in this paper indicates that in this region the self-similar problem must have either more than one solution or none in the neighbourhood of the arc $PE$. Indeed, the explicit solution constructed in [4] indicates that, using the fast evolutionary SW $A \to M_1$, one can, as before, construct a I-solution in the region above the segment $PE$ in the $(u_1, u_2)$ plane phase (Fig. 5a) up to the points of the arc $PE$ itself. Below this arc the solution has another structure (a II-solution). It contains a fast Jouguet SW (according to the post-jump state, $W = c_2^+$) $A \to J$, after which there will be a fast Riemann wave, yet another fast Jouguet SW (according to the pre-jump state, $W = c_2^-$), and then a slow SW. Because of the inclusion of a Riemann wave in the sequence, the initial state for the final slow SW will not belong to the shock adiabat ($A$), and therefore the final state will also be unrelated to the shock adiabat ($A$). This explains the fact that the II-solution continues over the arc $PE$ into a region on its other side, where there is already a I-solution.

The non-uniqueness region in the $(u_1, u_2)$ plane is bounded by the segments $PE$ of the shock adiabat ($A$) and $QP$ of the shock adiabat ($Q$) (the state $Q$ is defined by the condition $W_Q = W_J$) and then by integral curves of slow Riemann waves (shown hatched in Fig. 5a).

As observed in Section 2, under these conditions part of the evolutionary segment of the shock adiabat $QE$ adjoining the non-evolutionary segment at the Jouguet point $E$ ($W_E = c_1^+$) entered the non-uniqueness region. It is obvious from the diagram of Fig. 4(a) that the non-uniqueness region exists only when the point $E$ lies to the right of $J$. The Jouguet point may occupy such a position when $g \leqslant |\varkappa|(U_1^2 + U_2^2)$, that is when the anisotropy is small or the initial state has a large deformation. There is then a velocity range in which non-evolutionary $\{1\}$ discontinuities and one fast evolutionary SW, propagating at the same velocity $W$, exist.

The evolution diagram for media with $\varkappa < 0$ is shown in Fig. 4(b). Non-evolutionary $\{1\}$ discontinuities are represented by the two segments $FE$ and $DD'$ of the shock adiabat ($A$). In the hatched evolutionary

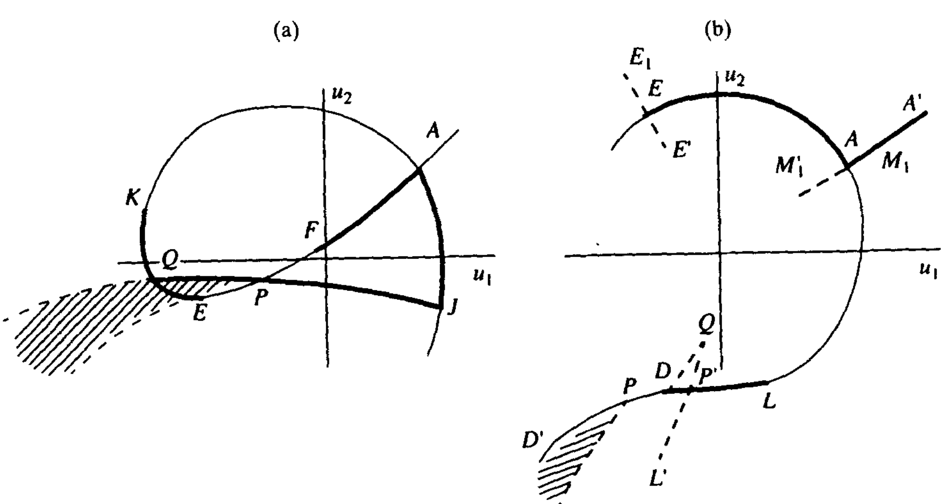


Fig. 5

rectangle above them there are segments $KE$ and $AA'$ of the shock adiabat ($A$), corresponding to two types of fast evolutionary SWs with the same velocity $W$ as the aforementioned non-evolutionary waves (corresponding to the points on a common vertical in Fig. 4).

According to the criterion proposed here, the presence of a non-evolutionary segment $FE$ should not lead to the appearance of non-uniqueness. The same may be said of the arc $DP$ of the lower branch of the shock adiabat ($A$). But to the right of $P$, for the non-evolutionary segment $PD'$ of the lower branch, the upper rectangle contains only one evolutionary fast SW with the same velocity $W$. Hence there must be a non-uniqueness region in the $(u_1, u_2)$ plane in the neighbourhood of the arc $PD'$.

Indeed, unlike the case $\varkappa > 0$, for media with $\varkappa < 0$, for a given value of $W$, there are two evolutionary slow SWs, thus producing two pairs of different solutions, acting in the $(u_1, u_2)$ plane in neighbourhoods of the non-evolutionary segments $FE$ and $DD'$ of the shock adiabat ($A$), on different sides of each of these segments. When the velocities of the fast and slow SWs in each of the pairs are the same, this results in the appearance of two non-evolutionary jumps – one in state $FE$, the other in $PD$.

When the velocity $W$ of the SW exceeds $W_E$, only one evolutionary fast SW remains. After this wave there may again be only one of the slow SWs. For that SW there are well-defined pre- and post-jump states. The pre-jump state is the state after the fast SW with the given velocity $W$, while the post-jump state is that after the non-evolutionary SW with the same velocity. Investigation shows that the slow SW corresponds in type to a point of the segment $LD$. Therefore, to the right of the vertical through the point $E$ in Fig. 4(b), one can construct a I-solution as a sequence of two SWs only on one side of the non-evolutionary {1} segment of the shock adiabat ($A$) in the $(u_1, u_2)$ plane. Direct solution of the self-similar problem shows that in the II-solution the fast SW $A \to M_1$ is followed by a slow Jouguet SW in state $E_1$ – a Jouguet point ($W = c_1^+$) of the shock adiabat ($M_1$), after that a slow Riemann wave, and then a slow Jouguet SW whose initial state does not belong to the shock adiabat ($A$), since these waves propagate at different velocities. The region of action of this solution is not bounded by the arc $PD'$. It has been shown [4] that in the $(u_1, u_2)$ plane the II-solution acts on both sides of the arc $PD'$, thereby producing a non-uniqueness region, in accordance with our criterion. This region is bounded on one side by the non-evolutionary branch $PD'$ of the shock adiabat ($A$) and on the other by Jouguet points of slow SWs ($W = c_1^-$) propagating through state $EE_1$ (the hatched zone in Fig. 5b).

When the Jouguet point $E$, where $W_E = c_1^+$, lies to the left of the vertical $W = c_2^-$ in the diagram, the region of existence of the II-solution, as it is obvious by continuity, contains part of the evolutionary section of the shock adiabat ($A$), which is situated to the left of $D$. The evolution diagram for that situation is shown in Fig. 4(b) by the dashed curve. It has been shown [4] that the non-uniqueness region extends along the shock adiabat up to the point $P'$, where $W = W_E$. In that region both the I- and II-solutions in the $(u_1, u_2)$ plane consist of the same wave sequences, but in the first, instead of the SW $A \to M_1$, there is a fast Riemann wave to state $M'_1$. The region of action of the II-solution is situated on both sides of the arc $P'D$, producing non-uniqueness. The non-uniqueness region in the $(u_1, u_2)$ plane for this case is bounded by the segment $DD'$ of the shock adiabat ($A$) and by curves consisting of Jouguet points for the pre-SW state for jumps from the initial states $AM'_1$ and $EE'_1$. The conservation laws do not forbid disintegration of the evolutionary SW corresponding to the segment $P'D$ into a system of II-solution waves.

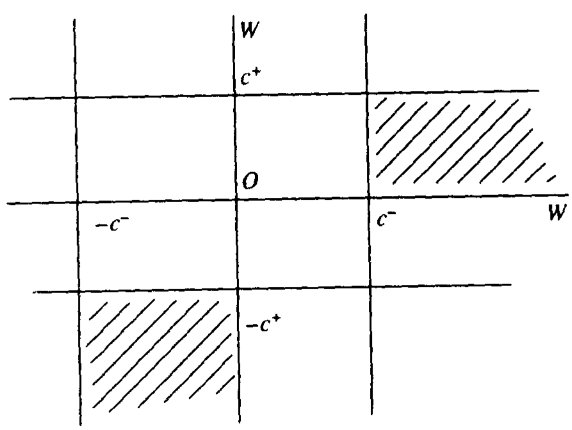


Fig. 6

For media with $\varkappa < 0$, the solution of the self-similar problem is always non-unique, but the position of the non-uniqueness region in the $(u_1, u_2)$ plane approaches the origin when the anisotropy increases (or, equivalently, when the initial deformation $U_t$ decreases).

It is obvious that the qualitative picture of the position of shock adiabats in the evolution diagram, and with it our conclusions regarding non-unique solvability, may remain valid for finite-amplitude waves also; for such waves the behaviour of solutions in the phase $u_i$-space has not been investigated, but the criterion presented in Section 2 is applicable.

For effective application of the criterion to a continuous medium, the medium must allow at least two types of perturbation to propagate in one direction. The criterion is not applicable for media in which one family of characteristics propagate in each direction. An evolution diagram for such a case is shown in Fig. 6. The rectangles corresponding to evolutionary discontinuities are shown hatched. There is clearly no non-evolutionary $\{1\}$ discontinuity corresponding to the rectangle $W \geqslant c^-$, $c^+ \leqslant W \leqslant 0$, since only the first and third quadrants in the diagram are physically meaningful. Non-uniqueness conditions for problems of gas dynamics and for longitudinal waves of the non-linear theory of elasticity, when only one family of characteristics propagates in each direction, have been obtained previously [11].

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